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ON THE CUBIC SHIMURA LIFT FOR PGL_3

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ABSTRACT

Two types of evidence are presented for the conjecture that the image of the cubic Shimura correspondence for PGL_3 is characterized by the nonvanishing of a certain period. Specifically, if a cusp form occurs in the pullback of the automorphic theta function on SO_8 under the adjoint homomorphism $\mathrm{PGL}_3 \to \mathrm{SO}_8$, then the cusp form is conjectured to be a Shimura lift.

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Introduction

The classical Shimura correspondence relates automorphic representations of the metaplectic double cover of SL_2 to automorphic representations π of PGL_2. According to the results of Waldspurger [Wa], the image of the correspondence may be characterized by either the nonvanishing of the central L-value $L(\frac{1}{2},\pi,\chi)$ for some quadratic character χ, or equivalently, the nonvanishing of a certain period, namely the integral of the automorphic form over a cycle.

Higher order Shimura correspondences also exist (see Flicker and Kazhdan [FK]). One would like similar characterizations of the lifts. L-function characterizations have proved elusive but there is one example in the literature of a period characterization for the cubic correspondence. To describe this, let F be a global field containing 3 distinct cube roots of unity, and $\mathbb{A}$ denote the adeles of F. Let $\mathrm{sym}_3\colon \mathrm{SL}_2 \to \mathrm{Sp}_4$ be the symmetric cube homomorphism, and let θ_{Sp_4} be the (classical) automorphic Weil representation of the double cover of $\mathrm{Sp}_4(\mathbb{A})$. This cover splits under the pullback along sym_3.

SL_2 Period Theorem: *Let π be an irreducible cuspidal representation of $\mathrm{SL}_2(\mathbb{A})$. Then the period*

$$\int_{\mathrm{SL}_2(F)\backslash \mathrm{SL}_2(\mathbb{A})} \varphi(g)\,\theta(\mathrm{sym}_3(g))\,dg$$

is nonzero for some φ in the space of π and some θ in the space of θ_{Sp_4} if and only if π is in the image of the cubic Shimura correspondence.

This result was proved by Ginzburg, Rallis and Soudry [GRS2] using a theta correspondence on G_2, and by Mao and Rallis [MR] using a relative trace formula.

The Shimura correspondence for the 3-fold cover $\widetilde{\mathrm{SL}}_3$ has as its image automorphic representations of PGL_3, not SL_3. This is in contrast with the rank one case, where the correspondence is from $\widetilde{\mathrm{SL}}_2$ to SL_2. The reason for this difference will be reviewed in Section 1.

Let F be a global field containing 3 distinct cube roots of unity, and $\mathbb{A}$ denote the adeles of F. In [GRS1] the theta representation θ_{SO_8} of $\mathrm{SO}_8(\mathbb{A})$ was introduced as a residue of a certain Eisenstein series. Let $\mathrm{Ad}\colon \mathrm{PGL}_3 \to \mathrm{SO}_8$ denote the adjoint representation. In this paper we make the

PGL_3 Period Conjecture: *Let π be an irreducible cuspidal representation of $\mathrm{PGL}_3(\mathbb{A})$. Then the period*

$$\int_{\mathrm{PGL}_3(F)\backslash \mathrm{PGL}_3(\mathbb{A})} \varphi(g)\,\theta(\,\mathrm{Ad}(g))\,dg$$

is nonzero for some φ in the space of π and some function θ in the space of θ_{SO_8} if and only if π is in the image of the cubic Shimura correspondence.

We will give two types of evidence for this conjecture. The evidence comes from finite fields (Section 1) and from global fields (Section 2).

In Section 1, we consider the minimal representation of $\mathrm{SO}_8(\mathbb{F}_q)$, which is the nontrivial representation of minimal Gelfand–Kirillov dimension. Its character is unipotent and of degree $q(q^2+1)^2$. We consider the pullback of this representation to $\mathrm{PGL}_3(\mathbb{F}_q)$ via Ad, and confirm that the irreducible characters which occur are in the image of the finite field cubic Shimura correspondence. To validate this heuristic, we show that it correctly predicts the SL_2 period result. Then we show that it also predicts the PGL_3 period conjecture.

In Section 2, we consider the PGL_3 period integral in the case where the cuspidal representation is replaced by an Eisenstein series involving a $\mathrm{GL}_2(\mathbb{A})$ cusp form. The integral in this case is divergent but may be renormalized in an obvious way. Formally, we unfold the PGL_3 integral and consider the resulting period. We show that the nonvanishing of the inner period is equivalent to the nonvanishing of a period of the $\mathrm{SL}_2(\mathbb{A})$ cusp form. Then using the SL_2 Period Theorem cited above, the nonvanishing of this period is equivalent to the form being in the image of the Shimura correspondence. To make this comparison, we establish a new identity between theta functions on Sp_4 and Sp_8; this identity (Theorem 2.1) is of interest in its own right.

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1. Evidence from finite fields

The n-fold Shimura correspondence is a functorial lifting of automorphic forms from the n-fold cover $\widetilde{\mathrm{GL}}_r$ of GL_r to GL_r. If r is a multiple of n, this may be regarded as a lift from $\widetilde{\mathrm{SL}}_r$ to PGL_r. If on the contrary r is prime to n, the Shimura correspondence may be regarded as a lift from $\widetilde{\mathrm{SL}_r}$ to SL_r. This dichotomy may be observed in the results of Savin [Sa]. Savin found that if G is a semisimple algebraic group over a nonarchimedean local field F containing the n-th roots of unity, and if d is the exponent of the fundamental group of the universal covering group of G (as an algebraic group), then the genuine part of the Iwahori Hecke algebra of the n-fold metaplectic cover $\widetilde{G}$ is isomorphic to the Iwahori Hecke algebra of G if $(n,d)=1$, while if $n|d$ it is isomorphic to the Iwahori Hecke algebra of the dual group of G.

Our interest in this paper will be with the special case $n=3$. We can concretely see why the cubic Shimura correspondence relates automorphic forms on $\widetilde{\mathrm{SL}}_3$ to automorphic forms on PGL_3 with a naive heuristic. To this end, we recall that the principal series representations (those induced from the Borel subgroup) are parametrized by characters of a maximal split torus (modulo the action of the Weyl group), and that, on principal series representations, the Shimura correspondence amounts to the n-th power maps on characters.

With this in mind, we can see why the Shimura correspondence effectively takes representations of $\widetilde{\mathrm{SL}}_3$ to representations of PGL_3. Let F be a local field. If χ_1,χ_2,χ_3 are characters of $F^\times$, the local Shimura correspondence sends the principal series $\tilde{\pi}(\chi_1,\chi_2,\chi_3)$ of $\widetilde{\mathrm{GL}}_3(F)$ to the representation $\pi(\chi_1^3,\chi_2^3,\chi_3^3)$ of $\mathrm{GL}_3(F)$. Since the central character of this representation is a cube, we can twist by $(\chi_1\chi_2\chi_3)^{-1}$ to obtain the representation $\pi(\chi_1^2\chi_2^{-1}\chi_3^{-1},\chi_2^2\chi_1^{-1}\chi_3^{-1},\chi_3^2\chi_1^{-1}\chi_2^{-1})$ of $\mathrm{PGL}_3(F)$. Moreover, this representation is unchanged if we multiply χ_1, χ_2 and χ_3 by a fixed character μ, which is as it should be because the representations $\tilde{\pi}(\chi_1,\chi_2,\chi_3)$ and $\tilde{\pi}(\mu\chi_1,\mu\chi_2,\mu\chi_3)$ of $\widetilde{\mathrm{GL}}_3(F)$ have the same restriction to $\widetilde{\mathrm{SL}}_3(F)$. In this way we are able to obtain a well-defined correspondence of representations from $\widetilde{\mathrm{SL}}_3(F)$ to $\mathrm{PGL}_3(F)$, at least for the principal series. Note that this does not work on SL_2.

The Shimura correspondence has an analog for representations of $\mathrm{GL}_r(F)$ when F is a finite field. In the finite field case, if the characteristic of F is prime to n, the cocycle describing the n-fold metaplectic cover of $\mathrm{GL}_r(F)$ splits and there is no need to distinguish between $\mathrm{GL}_r(F)$ and $\widetilde{\mathrm{GL}}_r(F)$. The n-th order Shimura correspondence still should correspond to the n-th power map on induction data. It is not enough just to consider principal series representations, however. Macdonald conjectured and Deligne and Lusztig proved [DL] that there is a natural parametrization of most irreducible representations of finite groups of Lie type by characters of maximal tori. For the split maximal tori, this agrees with the construction of the principal series representations by parabolic induction. In general, if T is a maximal torus and ν is a character of T (that is, of the group of F-rational points of T) then Deligne and Lusztig defined a generalized character $R_{T,\nu}$.

We only care about representations which are in "general position," that is, we wish to define the correspondence on Deligne–Lusztig characters $R_{T,\nu}$, where T is a maximal torus of $\mathrm{GL}_r(F)$, ν is a character of $T(F)$, and where we are willing to impose a finite number of conditions on ν which will guarantee that $R_{T,\nu}$ (or its negative) is an irreducible character. The Shimura correspondence should map the Deligne–Lusztig character $R_{T,\nu}$ parametrized by θ to R_{T,ν^n}. This definition will require modification when the source representation is of $\mathrm{SL}_r(F)$, but the particular modification is (as we have seen) going to be slightly different for the two cases $r = 2$ and $r = 3$.

This is, of course, just a formal definition, but let us show that it has some predictive value for periods of automorphic forms. We will show that this heuristic predicts the SL_2 Period Theorem of [GRS2] and [MR] in the introduction.

This result is consistent with a finite field calculation, which we may show as follows. Let $F = \mathbb{F}_q$ be a finite field with q elements, where we assume that $q \equiv 1$ mod 3. We embed $\mathrm{SL}_2(F)$ into $\mathrm{Sp}_4(F)$ via the symmetric cube homomorphism. We pull back the theta representation from $\mathrm{Sp}_4(F)$ to $\mathrm{SL}_2(F)$ and consider its decomposition into irreducibles. As with GL_2 we define the Shimura correspondence to be a map of Deligne–Lusztig characters $R_{T,\nu}$ in general position, where T is a maximal torus of SL_2 and ν a character of $T(F)$ in general position, namely we map $R_{T,\nu} \to R_{T,\nu^3}$.

There are two choices of maximal torus T of $\mathrm{SL}_2(F)$. The maximal split torus T_s, consisting of diagonal elements, has order $q-1$ and may be identified with $\mathbb{F}_q^\times$. The corresponding Deligne–Lusztig characters coincide with the principal series.

The maximal anisotropic torus T_a has order $q+1$ and may be identified with the group of norm 1 elements of $\mathbb{F}_{q^2}$. The corresponding Deligne–Lusztig characters coincide with the cuspidal representations.

Let λ_2: $\mathrm{SL}_2 \to \mathrm{Sp}_4$ be the symmetric cube embedding. The theta representation of Sp_4 is a representation of degree q^2 which has a model as an action on the space of functions on F^2. We use the notation of Srinivasan [Sr] for both characters and conjugacy classes of Sp_4. In her notation, the theta representation is $\theta_3 - \theta_7$. Here θ_3 corresponds to the Weil representation on the even dimensional functions on $\mathbb{F}_{q^2}$, and $-\theta_7$ is the Weil representation on the odd dimensional functions. (In her notation it is $-\theta_7$ rather than θ_7 which is a character.) Let us denote $\theta_3 - \theta_7$ by θ.

THEOREM 1.1: *Let $F = \mathbb{F}_q$ be a finite field, where $q \equiv 1$ modulo 3.*
(1) *A principal series representation $R_{T,\nu}$ of $\mathrm{SL}_2(F)$ in general position occurs in $\theta \circ \lambda_2$ if and only if ν is a cube, that is, if and only if it is a Shimura lift. The multiplicity (when it occurs) is* 3.
(2) *Every cuspidal representation in general position of $\mathrm{SL}_2(F)$ occurs with*

multiplicity 1 *in* $\theta \circ \lambda_2$, *and every cuspidal representation in general position is a Shimura lift.*

Proof: Since we are assuming that $q \equiv 1$ modulo 3, $3|q-1$ but $(3, q+1) = 1$. Thus every character of $T_a(F)$ is a cube, but not every character of $T_s(F)$ is a cube. Thus every cuspidal representation of SL_2 (parametrized by the characters of $T_a(F)$) is in the image of the cubic Shimura correspondence. On the other hand, only the principal series representations parametrized by characters which are cubes are in the image of the cubic Shimura correspondence. These are the characters χ of $T_s(F)$ which are trivial on

$$\begin{pmatrix} \rho & \\ & \rho^2 \end{pmatrix},$$

where ρ is a nontrivial cube root of unity in $F^\times$. This is consistent with results of Flicker [Fl] in the local case, where the Shimura correspondence is surjective on the discrete series but not on the principal series.

Let ν be a nontrivial character of the group of elements in $\mathbb{F}_{q^2}$ of norm 1. Then $\chi = -R_{T,\nu}$ is an effective irreducible cuspidal representation of $\mathrm{SL}_2(\mathbb{F}_q)$. Here are the relevant character values.

With notations as in Srinivasan [Sr],

$$\theta\left(\lambda_2\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}\right) = \theta(A_1) = q^2, \quad \chi\begin{pmatrix} 1 & \\ & 1 \end{pmatrix} = q-1.$$

This conjugacy class contributes $q^2(q-1)$.

$$\theta\left(\lambda_2\begin{pmatrix} -1 & \\ & -1 \end{pmatrix}\right) = \theta(A_1') = 1, \quad \chi\begin{pmatrix} -1 & \\ & -1 \end{pmatrix} = \nu(-1)(q-1).$$

This conjugacy class contributes $\nu(-1)(q-1)$.

$$\theta\left(\lambda_2\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}\right) = \theta(A_{41}) = -\bar{\epsilon} + \tilde{\epsilon}, \quad \chi\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} = -1,$$

and with $a \in \mathbb{F}_q^\times$ a nonsquare:

$$\theta\left(\lambda_2\begin{pmatrix} 1 & a \\ & 1 \end{pmatrix}\right) = \theta(A_{42}) = -(-\bar{\epsilon} + \tilde{\epsilon}), \quad \chi\begin{pmatrix} 1 & a \\ & 1 \end{pmatrix} = -1.$$

The last two contributions cancel.

$$\theta\left(\lambda_2\begin{pmatrix} -1 & -1 \\ & -1 \end{pmatrix}\right) = \theta(A_{41}') = -\bar{\epsilon} - \tilde{\epsilon} = 1, \quad \chi\begin{pmatrix} -1 & -1 \\ & -1 \end{pmatrix} = -\nu(-1).$$

$$\theta\left(\lambda_2\begin{pmatrix} -1 & -a \\ & -1 \end{pmatrix}\right) = \theta(A_{42}') = -\bar{\epsilon} - \tilde{\epsilon} = 1, \quad \chi\begin{pmatrix} -1 & -a \\ & -1 \end{pmatrix} = -\nu(-1).$$

The last two conjugacy classes together account for q^2-1 group elements and they contribute $-(q^2-1)\nu(-1)$. The values along the noncentral elements of the split torus may be discarded since χ vanishes. Finally, we have the contribution of the elliptic conjugacy classes. Let $\zeta \in \mathbb{F}_{q^2}$ have norm 1. Assume that $\zeta \neq \pm 1$. Then there is a conjugacy class in SL_2 with eigenvalues ζ, ζ^{-1}. This maps to a conjugacy class $B_4(3i,i)$ in Srinivasan's notation, or rarely (if $\zeta^4=1$) $B_6(i)$. In either case, $\theta = 1$ and $\chi = -(\nu(\zeta)+\nu(\zeta)^{-1})$. There are $q(q-1)$ elements in the conjugacy class. The elements ζ and ζ^{-1} parametrize the same conjugacy class. So we have a contribution of

$$-q(q-1) \sum_{\zeta\in\mathbb{F}_{q^2}, N(\zeta)=1, \zeta\neq\pm1} \nu(\zeta) = q(q-1)(1+\nu(-1)).$$

From these data, we find that the inner product of $\theta\circ\lambda_2$ with the cuspidal representation χ is 1, independent of ν.

Next let us consider the principal series representations. Let ν be a character of $T_s(F) = \mathbb{F}_q^\times$. The conjugacy class in Sp_4 of

$$\lambda_2\left(\begin{pmatrix} t & \\ & t^{-1}\end{pmatrix}\right), \quad t \neq \pm 1$$

depends upon t. These conjugacy classes did not contribute in the cuspidal case, since the cuspidal character vanished. Now, however, we need to compute them. We find that the conjugacy class of this matrix (in Srinivasans's notation) is either $B_3(3i,i)$ or $C_3'(i)$ *unless* $t=\rho$ or ρ^2, in which case it is $C_3(i)$. The value of θ is thus

$$\begin{cases} 1 & \text{if } t\neq\rho,\ \rho^2, \\ q & \text{if } t=\rho \text{ or } \rho^2. \end{cases}$$

The number of elements of each conjugacy class is $q(q+1)$, and the value of χ is $\nu(t)+\nu(t^{-1})$. Note that

$$\begin{pmatrix} t & \\ & t^{-1}\end{pmatrix}, \quad \begin{pmatrix} t^{-1} & \\ & t\end{pmatrix}$$

actually lie in the same conjugacy class, so we may sum over all values of t and then divide by two, or (equivalently) we may sum over all values of t and replace χ by $\nu(t)$. Thus the total contribution of these terms is

$$q(q+1)\left[\sum_{t\neq\pm1,\rho,\rho^2} \nu(t) + q(\nu(\rho)+\nu(\rho^2))\right]$$
$$= q(q+1)[(q-1)\big(\nu(\rho)+\nu(\rho^2)\big) - \nu(1) - \nu(-1)].$$

The contribution of the identity is $(q+1)q^2$, similar to the cuspidal case except that the character value is now $q+1$.

The contribution of $-I$ is $\nu(-1)(q+1)$.

The contribution of the unipotent conjugacy classes cancels, and the contribution of the negative unipotent conjugacy classes equals $\nu(-1)(q^2-1)$.

Summing and dividing by the order of the group, we find that the multiplicity of the principal series in $\theta \circ \lambda_2$ is $1 + \nu(\rho) + \nu(\rho^2)$. Thus the principal series representation occurs with multiplicity 3 if ν is a cube, zero otherwise. ∎

To conclude, the inner product of a character of $\mathrm{SL}_2(F)$ with the pullback of the theta function on Sp_4 to SL_2 is nonzero if and only if the representation is in the image of the Shimura correspondence. This is always the case for the cuspidal representations, but not always for the principal series. The analogy with the global result of Ginzburg, Soudry, Rallis and of Mao and Rallis is clear. So for SL_2, a heuristic based on the finite field case is capable of predicting a period result for automorphic forms.

Now let us turn to the case of the Shimura correspondence from $\widetilde{\mathrm{SL}}_3$ to PGL_3. It will be convenient to assume that $q \equiv 1$ modulo 6.

Let T_1 be a maximal torus of SL_3. Assume that $T_1 = \mathrm{SL}_3 \cap T$, where T is a maximal torus of GL_3. Let Z be the center of GL_3, which is of course contained in T. We regard the determinant as a map from $\mathrm{GL}_3 \to Z \subset T$. Then if ν is a character of $T_1(F)$, we may extend it to $T(F)$ and form the Deligne–Lusztig character $R_{T,\nu}$, which we tensor with the composition $(\nu \circ \det)^{-1}$.

It is easily checked that the resulting character $R_{T,\nu} \otimes (\nu \circ \det)^{-1}$ does not depend on the extension of ν from $T_1(F)$ to $T(F)$. Moreover, this representation has trivial central character, so it may be regarded as a character of $\mathrm{PGL}_3(F)$. Thus we define the Shimura correspondence, at least for those Deligne–Lusztig characters which are irreducible (or the negative of an irreducible) representations of SL_3.

Let λ_3: $\mathrm{PGL}_3(F) \to \mathrm{SO}_8(F)$ denote the adjoint homomorphism. The theta representation θ of $\mathrm{SO}_8(F)$, that is, the nontrivial representation of minimal Gelfand–Kirillov dimension, is unipotent, and is thus luckily one which is tabulated by Geck and Pfeiffer [GP]. Indeed, it is the representation which they denote χ_2. We claim that (with the Shimura correspondence defined naively as above in the finite field case) a representation of PGL_3 is a lift if and only if it occurs in the pullback of θ. This will therefore be evidence for a global conjecture.

THEOREM 1.2: *Let $F = \mathbb{F}_q$ be a finite field, where $q \equiv 1$ modulo 6.*

(1) *A principal series representation $R_{T,\nu}$ of $\mathrm{PGL}_3(F)$ in general position occurs in $\theta \circ \lambda_3$ if and only if ν is a cube, that is, if and only if it is a Shimura lift. The multiplicity (when it occurs) is* 3.
(2) *Every cuspidal representation in general position of $\mathrm{PGL}_3(F)$ occurs with multiplicity* 1 *in $\theta \circ \lambda_3$, and every cuspidal representation in general position is a Shimura lift.*

Proof: First let us show that as in the case of SL_2 the cuspidal representations are all Shimura lifts—but for a different reason. Suppose that $T = T_a$ is a maximal anisotropic torus of GL_3, so that $T(\mathbb{F}_q) \cong \mathbb{F}_{q^3}^\times$. Let $N\colon \mathbb{F}_{q^3}^\times \to \mathbb{F}_q^\times \subset \mathbb{F}_{q^3}^\times$ be the norm map, which we regard as a map $T(F) \to T(F)$. If ν is a character of $T_1(F)$, we extend ν to $T(F)$, then form the character τ of $T(F)$, where $\tau(x) = \nu(x^3/N(x))$. Then the Shimura lift of $R_{T,\nu}$ is $R_{T,\tau}$. This is equivalent to the previous definition.

To show that every cuspidal character of $\mathrm{PGL}_3(F)$ is of this form, what we must show is that every character τ of $T(F)$ whose restriction to $Z(F)$ is trivial is of this form. Dualizing, we must prove the exactness of

$$Z(F) \to T(F) \to T(F) \quad \text{or} \quad \mathbb{F}_q^\times \to \mathbb{F}_{q^3}^\times \to \mathbb{F}_{q^3}^\times,$$

where the first map is the inclusion and the second map sends $x \to x^3/N(x)$. Indeed, if $x \to x'$ denotes the Frobenius map in $\mathrm{Gal}(\mathbb{F}_{q^3}/\mathbb{F}_q)$, $x^3/N(x) = 1$ is equivalent to $x/x' = x''/x = (x/x')'$, that is (by Galois theory) to $x \in \mathbb{F}_q^\times$. This concludes the proof that the Shimura correspondence as we've defined it is surjective on cuspidal representations.

We will check this for the principal series and for the cuspidal representations in general position. Notation will be as in Steinberg [St] for conjugacy classes of GL_3, and for Geck and Pfeiffer [GP] for conjugacy classes of SO_8. To confirm Theorem 1.2, we must tabulate which conjugacy classes of GL_3 map to which conjugacy classes of SO_8.

We pull the theta representation of SO_8 back to GL_3 and compute its inner product with a fixed representation of the latter group.

Let $\chi = \pi(\chi_1, \chi_2, \chi_3)$ be a principal series representation of GL_3 with central trivial character $\chi_1\chi_2\chi_3 = 1$. We will assume that the characters χ_i are in general position. Specifically we assume $\chi_i^2 \neq 1$ and that $\chi_i^2 \neq \chi_j^2$.

We write

$$A_1 = \begin{pmatrix} a & & \\ & a & \\ & & a \end{pmatrix} \mapsto s_{51}.$$

Here A_1 is Steinberg's notation for the conjugacy class of GL_3, and s_{51} is the notation of Geck and Pfeiffer for this conjugacy class of SO_8. For this $\chi = (q+1)(q^2+q+1)$, $\theta = q(q^2+1)^2$, and there are $q-1$ conjugacy classes of this type, each with 1 element.

$$A_2 = \begin{pmatrix} a & 1 & \\ & a & \\ & & a \end{pmatrix} \mapsto u_6.$$

For this $\chi = 2q+1$, $\theta = q$. There are $q-1$ conjugacy classes of this type and each has $(q^2-1)(q^2+q+1)$ elements.

$$A_3 = \begin{pmatrix} a & 1 & \\ & a & 1 \\ & & a \end{pmatrix} \mapsto u_{12}.$$

For this $\chi = 1$, $\theta = q$. There are $q-1$ conjugacy classes of this type and each has $q(q-1)^2(q+1)(q^2+q+1)$ elements.

Assuming $a \neq b$,

$$A_4 = \begin{pmatrix} a & & \\ & a & \\ & & b \end{pmatrix} \mapsto \begin{cases} s_{45} & \text{if } a \neq -b, \\ s_{47} & \text{if } a = -b. \end{cases}$$

In this case

$$\chi = (q+1)[\chi_1\chi_2(a)\chi_3(b) + \chi_2\chi_3(a)\chi_1(b) + \chi_3\chi_1(a)\chi_2(b)]$$

and

$$\theta = \begin{cases} 3q+1 & \text{if } a \neq -b, \\ 4q & \text{if } a = -b. \end{cases}$$

There are $q^2(q^2+q+1)$ elements in each conjugacy class.

Assuming $a \neq b$,

$$A_5 = \begin{pmatrix} a & 1 & \\ & a & \\ & & b \end{pmatrix} \mapsto \begin{cases} s_{45}u_6 & \text{if } a \neq -b, \\ s_{47}u_6 & \text{if } a = -b. \end{cases}$$

In the first case, the unipotent is $(2,2,2)$ in the notation of Geck and Pfeiffer and in the second it doesn't matter. We have

$$\chi = \chi_1\chi_2(a)\chi_3(b) + \chi_2\chi_3(a)\chi_1(b) + \chi_3\chi_1(a)\chi_2(b)$$

and

$$\theta = \begin{cases} 1 & \text{if } a \neq -b, \\ q & \text{if } a = -b. \end{cases}$$

There are $q^2(q^2-1)(q^2+q+1)$ elements in each conjugacy class.

Parametrize the conjugacy class of $\mathrm{diag}(a/b, b/c, c/a, b/a, c/b, a/c)$ in SO_6 by a partition. Thus if 2 of the eigenvalues are equal but the other four distinct, the partition is $(2,1,1,1,1)$. For the partitions $(1,1,1,1,1,1)$ and $(2,1,1,1,1)$, which may be combined,

$$A_6 = \begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix} \mapsto \begin{cases} s_1 & (1,1,1,1,1,1) \text{ or } (2,1,1,1,1), \\ s_{14} & (2,2,1,1) \text{ or } (2,2,2), \\ s_{22} & (3,3). \end{cases}$$

Then

$$\chi = \chi_1(a)\chi_2(b)\chi_3(c) + \cdots$$

(6 terms) and

$$\theta = \begin{cases} 4 & (1,1,1,1,1,1) \text{ or } (2,1,1,1,1), \\ q+3 & (2,2,1,1) \text{ or } (2,2,2), \\ q^2+q+2 & (3,3). \end{cases}$$

There are $q^3(q+1)(q^2+q+1)$ elements in each class.

To calculate the A_4 and A_5 contributions we observe that

$$\sum_{\substack{a,b \\ a\neq\pm b}} \chi_1\chi_2(a)\chi_3(b) = \sum_a \chi_1\chi_2(a)[-\chi_3(a) - \chi_3(-a)] = -(q-1)\big(1+\chi_3(-1)\big),$$

while

$$\sum_{\substack{a,b \\ a=-b}} \chi_1\chi_2(a)\chi_3(b) = (q-1)\chi_3(-1).$$

Now we can compute the A_4 contribution:

$$\begin{aligned} &(q+1)q^2(q^2+q+1)[-(3q+1)(q-1)(3+\chi_1(-1)+\chi_2(-1)+\chi_3(-1)) \\ &\qquad\qquad + (q-1)(4q)(\chi_1(-1)+\chi_2(-1)+\chi_3(-1))] \\ &= q^2(q^2-1)(q^2+q+1)[-(9q+3)+(q-1)(\chi_1(-1)+\chi_2(-1)+\chi_3(-1))]. \end{aligned}$$

The A_5 contribution is

$$q^2(q^2-1)(q-1)(q^2+q+1)[-3+(q-1)(\chi_1(-1)+\chi_2(-1)+\chi_3(-1))].$$

The A_6 contribution breaks into three pieces. To evaluate the first piece, we must compute the sum

$$\sum_{b\neq 1} \sum_{\substack{c\neq 1,b,b^{-1},b^2 \\ c^2\neq b}} \chi_2(b)\chi_3(c).$$

Write this as $I_1 + I_2$, where

$$I_1 = \sum_{b \notin F^{\times 2}} \chi_2(b) \sum_{c \neq 1, b, b^{-1}, b^2} \chi_3(c) \quad \text{and} \quad I_2 = \sum_{\substack{b \neq 1 \\ b \in F^{\times 2}}} \sum_{\substack{c \neq 1, b, b^{-1}, b^2 \\ c^2 \neq b}} \chi_2(b)\chi_3(c).$$

Here $F^{\times 2}$ denotes the subgroup of $F^\times$ consisting of squares. Since we assume that -1 is a square, it is not difficult to check that $I_1 = 0$, while

$$I_2 = (q-1)\big(5 + \chi_1(-1) + \chi_2(-1) + \chi_3(-1) + 2[\chi_2\chi_3^2(\rho) + \chi_2^2\chi_3(\rho)]\big).$$

This corresponds to the conjugacy class s_1 with character value $\theta = 4$. Note that (assuming $\chi_1\chi_2\chi_3 = 1$) this is symmetric in the χ_i.

The partition $(3,3)$ gives a contribution

$$(q-1)\big(\chi_2^2\chi_3(\rho) + \chi_2\chi_3^2(\rho)\big).$$

The conjugacy class is s_{22} with $\theta = 1$.

The partitions $(2,2,2)$ and $(2,2,1,1)$ give

$$\begin{aligned} &-(q-1)[3 + \chi_1(-1) + \chi_2(-1) + \chi_3(-1) \\ &\quad + \chi_1^{-1}\chi_3(\rho) + \chi_2^{-1}\chi_1(\rho) + \chi_3^{-1}\chi_2(\rho) + \chi_1\chi_3^{-1}(\rho) + \chi_2\chi_1^{-1}(\rho) + \chi_3\chi_2^{-1}(\rho)]. \end{aligned}$$

This conjugacy class is s_{14} with $\theta = q + 3$.

Summing the various contributions, using $\chi_3 = (\chi_1\chi_2)^{-1}$ and dividing by the order of the group gives

$$\frac{\left(1 + \chi_1(\rho)^2\,\chi_2(\rho) + \chi_1(\rho)\,\chi_2(\rho)^2\right) R}{(-1+q)^2\,\chi_1(\rho)^2\chi_2(\rho)^2}$$

where

$$R = (-3\,\chi_1(\rho) - q\,\chi_1(\rho) + 7\chi_2(\rho) + q^2\,\chi_2(\rho) - 3\chi_1(\rho)^2\,\chi_2(\rho)^2 - q\chi_1(\rho)^2\,\chi_2(\rho)^2).$$

Thus the contribution vanishes unless $\chi_1(\rho) = \chi_2(\rho)$, and making this assumption the inner product simplifies to 3.

The cuspidal case is similar, and we leave most of the details to the reader. The principal new feature is that we must include the C_1 conjugacy classes, which map to s_3 in the notation of Geck and Pfeiffer. The value of $\theta = 1$. There are $(q^3 - q)/3$ conjugacy classes of this type and each has $q^3(q-1)(q^2-1)$ elements in each conjugacy class. ∎

The significance which we attach to these two theorems is as follows. A comparison of Theorem 1.1 with the period result of Ginzburg, Rallis and Soudry and of Mao and Rallis shows that a heuristic based on finite field calculations can lead to a correct prediction. Theorem 1.2 suggests that we should expect an automorphic representation of PGL_3 to occur in the pullback of the theta representation of SO_8 if and only if the representation is in the image of the cubic Shimura correspondence.

2. Evidence from global fields

In this section, let F be a global field, and let $\mathbb{A}$ denote the adeles of F. We will assume that F contains three cube roots of unity and denote by $\rho \neq 1$ an element in F such that $\rho^3 = 1$.

Let $J = (J_{ij})$ denote the 8×8 matrix with $J_{ij} = 1$ if $i+j = 9$, $J_{ij} = 0$ otherwise, and let SO_8 denote the special orthogonal group stabilizing the quadratic form corresponding to J: $\mathrm{SO}_8 = \{g \in \mathrm{GL}_8 : {}^t gJg = J\}$.

The adjoint representation of GL_3 is a homomorphism from GL_3 to SO_8. We denote this map by Ad. To describe the image of this map we choose the following basis for the set of all 3×3 trace zero matrices. Denote by h_{ij} the 3×3 matrix whose (i,j)-entry is one and whose other entries are zero. The basis we choose is $e_1 = h_{13}$; $e_2 = h_{23}$; $e_3 = h_{12}$; $f_1 = h_{31}$; $f_2 = h_{32}$; $f_3 = h_{21}$; $e_4 = \frac{1}{3}((\rho - 1)h_{11} + (1-\rho^2)h_{22} + (\rho^2 - \rho)h_{33})$; and $f_4 = \frac{1}{3}((\rho^2 - 1)h_{11} + (1-\rho)h_{22} + (\rho - \rho^2)h_{33})$. Let I denote the 8×8 identity matrix, and let $e_{i,j}$ denote the 8×8 matrix whose (i,j)-entry is one and whose other entries are zero. Let $e'_{i,j} = e_{i,j} - e_{9-j,9-i}$. Then with respect to the basis $\{e_1, e_2, e_3, e_4, f_4, f_3, f_2, f_1\}$ of all 3×3 trace zero matrices we have

$$\mathrm{Ad}\begin{pmatrix} 1 & x & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix} = I + x(e'_{1,2} + e'_{3,4} + e'_{3,5}) - x^2 e_{3,6},$$

$$\mathrm{Ad}\begin{pmatrix} 1 & & \\ & 1 & y \\ & & 1 \end{pmatrix} = I + y(-e'_{1,3} + \rho^2 e'_{2,4} + \rho e'_{2,5}) - y^2 e_{2,7},$$

$$\mathrm{Ad}\begin{pmatrix} 1 & & z \\ & 1 & \\ & & 1 \end{pmatrix} = I - z(\rho e'_{1,4} + \rho^2 e'_{1,5} + e'_{2,6}) - z^2 e_{1,8},$$

$$\mathrm{Ad}(\mathrm{diag}(a,b,c)) = \mathrm{diag}(ac^{-1}, bc^{-1}, ab^{-1}, 1, 1, a^{-1}b, b^{-1}c, a^{-1}c).$$

In particular, the center of GL_3 is (of course) in the kernel of Ad.

Let π be an irreducible cuspidal representation of $\mathrm{PGL}_3(\mathbb{A})$. Let $\theta_{\mathrm{SO}_8}(\cdot)$ denote a vector in the theta representation of SO_8. In this section we shall give some global evidence that the period

$$\int_{\mathrm{PGL}_3(F)\backslash\mathrm{PGL}_3(\mathbb{A})} \varphi(g)\theta_{\mathrm{SO}_8}(\mathrm{Ad}(g))\,dg, \tag{2.1}$$

where φ is in the space of π, is related to the cubic Shimura lifting as described above. The idea is as follows. Let τ be a cusp form on $\mathrm{GL}_2(\mathbb{A})$. Let P denote the standard $(2,1)$ parabolic subgroup of PGL_3, and form the Eisenstein series $E_\tau(g,s)$ corresponding to the induced representation $\mathrm{Ind}_{P(\mathbb{A})}^{\mathrm{PGL}_3(\mathbb{A})}\ \tau\otimes\delta_P^s$. In this section, we formally replace $\varphi(g)$ in (2.1) by $E_\tau(g,s)$ and proceed with the usual unfolding as if the integral converges. If the integral (2.1) is to characterize the cubic Shimura lift, then one would expect that using the formal unfolding we would get an inner period which would characterize the Shimura cubic lifting for τ. We shall confirm this here.

Let $\varphi_\tau(h)$ denote a vector in the space of τ. It follows from [GRS2], p. 277, that a cusp form τ is in the image of the cubic Shimura lift for GL_2 if and only if the period

$$\int_{\mathrm{SL}_2(F)\backslash\,\mathrm{SL}_2(\mathbb{A})} \varphi_\tau(h)\widetilde{\theta}_{\mathrm{Sp}_4}^{\phi'}(\widetilde{s}^3(h))\,dh \tag{2.2}$$

is nonzero for some choice of data. Here $\phi'\in\mathcal{S}(\mathbb{A}^2)$, the Schwartz space of $\mathbb{A}^2$, and $\widetilde{\theta}_{\mathrm{Sp}_4}^{\phi'}$ is the corresponding theta function on $\widetilde{\mathrm{Sp}}_4(\mathbb{A})$, the double cover of $\mathrm{Sp}_4(\mathbb{A})$. Also, $\widetilde{s}^3(h)$ is the symmetric cube representation of SL_3 as described in [GRS2], p. 257. We will exhibit a relation between the inner period obtained from (2.1) when $\varphi(g)$ is formally replaced by $E_\tau(g,s)$ and the period (2.2). In fact, this inner period will converge absolutely. Our proof that it is equal to (2.2) is based on a new identity between theta functions on $\widetilde{\mathrm{Sp}}_4(\mathbb{A})$ and on $\widetilde{\mathrm{Sp}}_8(\mathbb{A})$ given in Theorem 2.1 below. We turn to the details.

Formally unfolding the integral

$$\int_{\mathrm{PGL}_3(F)\backslash\mathrm{PGL}_3(\mathbb{A})} E_\tau(g,s)\theta_{\mathrm{SO}_8}(\mathrm{Ad}(g))\,dg \tag{2.3}$$

we obtain as inner integration

(2.4)

$$\int_{\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A})} \varphi_\tau(h)\int_{(F\backslash\mathbb{A})^2}\theta_{\mathrm{SO}_8}\Big(\mathrm{Ad}\begin{pmatrix}1 & & z\\ & 1 & y\\ & & 1\end{pmatrix}\mathrm{Ad}\begin{pmatrix}h & \\ & 1\end{pmatrix}\Big)\,dy\,dz\,dh.$$

Expand the theta function along the highest root. In other words, let ψ be a nontrivial character of $F\backslash\mathbb{A}$, and write

$$\theta_{\mathrm{SO}_8}(m)=\sum_{\alpha\in F}\int_{F\backslash\mathbb{A}}\theta_{\mathrm{SO}_8}(x(r)m)\psi(\alpha r)\,dr$$

where $x(r) = I + re'_{1,7}$. The main contribution of this expansion to (2.4) comes from the nontrivial characters. Ignoring the contribution of the trivial orbit we consider the integral

$$\int_{\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A})} \varphi_\tau(h) \int_{(F\backslash\mathbb{A})^2} \sum_{\alpha\in F^\times} \int_{F\backslash\mathbb{A}} \theta_{\mathrm{SO}_8}\Big(x(r)\,\mathrm{Ad}\begin{pmatrix}1 & & z\\ & 1 & y\\ & & 1\end{pmatrix}\mathrm{Ad}\begin{pmatrix}h & \\ & 1\end{pmatrix}\Big)\psi(\alpha r)\,dr\,dy\,dz\,dh.$$

The group GL_2 acts on $x(r)$ by $g\colon x(r)\to x((\det g)r)$. Hence we may collapse the summation and integration to obtain as an inner integration

$$\int_{\mathrm{SL}_2(F)\backslash \mathrm{SL}_2(\mathbb{A})} \varphi_\tau(h) \int_{(F\backslash\mathbb{A})^3} \theta_{\mathrm{SO}_8}\Big(x(r)\,\mathrm{Ad}\begin{pmatrix}1 & & z\\ & 1 & y\\ & & 1\end{pmatrix}\mathrm{Ad}\begin{pmatrix}h & \\ & 1\end{pmatrix}\Big)\psi(r)\,dr\,dy\,dz\,dh. \tag{2.5}$$

Notice that this integral converges absolutely. We will now relate this integral to the integral (2.2).

Let Q be the maximal parabolic subgroup of SO_8 with Levi decomposition given by $Q = (\mathrm{GL}_2\times \mathrm{SO}_4)V$ and let $Q^0\subset Q$ be the group $Q^0 = (\mathrm{SL}_2\times\mathrm{SO}_4)V$. The unipotent group V is isomorphic to H_9, the Heisenberg group in nine variables. We will denote by $(\overline{x}|\overline{y}|z)$ an element of H_9 where $\overline{x}=(x_1,x_2,x_3,x_4)$ and $\overline{y}=(y_1,y_2,y_3,y_4)$ (see [GRS3], p. 185). Then the above isomorphism is given by

$$\begin{aligned} I + x_1e'_{2,3}+x_2e'_{2,4}+x_3e'_{2,5}+x_4e'_{2,6} &\mapsto (x_1,-x_2,-x_3,x_4|\overline{0}|0),\\ I + y_1e'_{1,3}+y_2e'_{1,4}+y_3e'_{1,5}+y_4e'_{1,6} &\mapsto (\overline{0}|y_1,-y_2,-y_3,y_4|0),\\ I+ze'_{1,7} &\mapsto (\overline{0}|\overline{0}|z). \end{aligned}$$

Let $\widetilde{\mathrm{Sp}}_8(\mathbb{A})$ denote the metaplectic double cover of $\mathrm{Sp}_8(\mathbb{A})$. When the cover splits over a subgroup of $\mathrm{Sp}_8(\mathbb{A})$, we shall regard the subgroup as embedded in $\widetilde{\mathrm{Sp}}_8(\mathbb{A})$. Let $\widetilde{\theta}_{\mathrm{Sp}_8}$ denote the theta representation of $\widetilde{\mathrm{Sp}}_8(\mathbb{A})$. Recall that $\widetilde{\theta}_{\mathrm{Sp}_8}$ is a representation of $H_9(\mathbb{A})\widetilde{\mathrm{Sp}}_8(\mathbb{A})$. As in the Theorem in Section 3 part (c) of [GRS2] (see also [GRS1] Section 4), a Fourier coefficient of $\theta_{\mathrm{SO}_8}(\cdot)$ may be realized as a theta function on $\widetilde{\mathrm{Sp}}_8$. More precisely, we have

$$\int_{F\backslash\mathbb{A}} \theta_{\mathrm{SO}_8}(x(r)vg)\psi(r)\,dr = \widetilde{\theta}^{\phi}_{\mathrm{Sp}_8}(vg) \tag{2.6}$$

where $v \in V(\mathbb{A}) \simeq H_9(\mathbb{A})$, $g \in \mathrm{SL}_2(\mathbb{A}) \times \mathrm{SO}_4(\mathbb{A})$, and $\phi \in \mathcal{S}(\mathbb{A}^4)$ is suitably chosen. The group $\mathrm{SL}_2 \times \mathrm{SO}_4$ is embedded in $\widetilde{\mathrm{Sp}}_8$ by the tensor product representation.

We use the identity (2.6) in (2.5). Notice that $\mathrm{Ad}\begin{pmatrix} 1 & & z \\ & 1 & y \\ & & 1 \end{pmatrix}$ and $\mathrm{Ad}\begin{pmatrix} h & \\ & 1 \end{pmatrix}$ give subgroups of Q^0. For $y \in \mathbb{A}$ let us define the following elements of $H_9(\mathbb{A})$: $\ell_1(y) = (0, -\rho^2 y, -\rho y, 0| - y, 0, 0, 0|0)$, $\ell_2(z) = (0,0,0,-z|0,\rho z, \rho^2 z, 0|0)$. Set $\ell(y,z) = \ell_1(y)\ell_2(z)$. Under the isomorphism of V with H_9 as described above we have $\mathrm{Ad}\begin{pmatrix} 1 & & z \\ & 1 & y \\ & & 1 \end{pmatrix} \mapsto \ell(y,z)$. For $h \in \mathrm{SL}_2$ denote by $i(h)$ the diagonal embedding of SL_2 in $\mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2$. Using (2.6) in (2.5) we then obtain

$$\int_{\mathrm{SL}_2(F)\backslash \mathrm{SL}_2(\mathbb{A})} \int_{(F\backslash\mathbb{A})^2} \varphi_\tau(h)\widetilde{\theta}^{\phi}_{\mathrm{Sp}_8}(\ell(y,z)i(h))\,dy\,dz\,dh. \tag{2.7}$$

Here we view $i(h)$ as a matrix of $\widetilde{\mathrm{Sp}}_8$ via the embedding of $\mathrm{SL}_2 \times \mathrm{SO}_4$ in $\widetilde{\mathrm{Sp}}_8$.

We claim that the integral (2.7) is related to the period (2.2). To show this, define the symmetric cube embedding of SL_2 in Sp_4 as follows:

$$s^3\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & (1-\rho^2)x^2 & 2x^3 \\ & 1 & -2\rho^2 x & (1-\rho^2)x^2 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix}\begin{pmatrix} 1 & (1-\rho)x & 0 & 0 \\ & 1 & 0 & 0 \\ & & 1 & (\rho-1)x \\ & & & 1 \end{pmatrix},$$

$$s^3\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} = \begin{pmatrix} & & & 1 \\ & & -\rho^2 & \\ & \rho & & \\ -1 & & & \end{pmatrix}.$$

This defines the symmetric cube embedding since these two matrices generate the group SL_2. Let ω_ψ^8 (resp. ω_ψ^4) denote the Weil representation of $\widetilde{\mathrm{Sp}}_8$ (resp. $\widetilde{\mathrm{Sp}}_4$). Given $\phi \in \mathcal{S}(\mathbb{A}^4)$ we define

$$\phi'(r_1, r_2) = \int_{\mathbb{A}^2} \omega_\psi^8(\ell(y,z))\phi(r_1, r_2, 0, 0)\,dy\,dz.$$

Using the action of the Weil representation (see [GRS3], p. 188) we obtain

$$\phi'(r_1,r_2) = \int_{\mathbb{A}^2} \phi(r_1, r_2 + \rho^2 y, \rho y, -z)\,\psi(\rho^2 r_2 z - yz)\,dy\,dz.$$

Hence $\phi' \in \mathcal{S}(\mathbb{A}^2)$.

The relation between (2.7) and (2.2) is then a consequence of the following result.

THEOREM 2.1: *The $\widetilde{\mathrm{Sp}}_8$ and $\widetilde{\mathrm{Sp}}_4$ theta functions are related by the identity*

$$\int_{(F\backslash\mathbb{A})^2} \widetilde{\theta}^{\phi}_{\mathrm{Sp}_8}(\ell(y,z)i(h))\,dy\,dz = \widetilde{\theta}^{\phi'}_{\mathrm{Sp}_4}(s^3(h)).$$

Proof: We unfold the left-hand side. It equals

$$\sum_{\xi_1,\xi_2,\xi_3,\xi_4\in F} \int_{(F\backslash\mathbb{A})^2} \omega^8_\psi(\ell(y,z)i(h))\phi(\xi_1,\xi_2,\xi_3,\xi_4)\,dy\,dz.$$

Using the action of the Weil representation, this equals

$$\sum_{\xi_1,\xi_2,\xi_3,\xi_4\in F} \int_{(F\backslash\mathbb{A})^2} \omega^8_\psi(\ell(\xi_3,\xi_4)\ell(y,z)i(h))\phi(\xi_1,\xi_2,0,0)\,dy\,dz.$$

Collapsing summation with integration we obtain

$$\sum_{\xi_1,\xi_2\in F} \int_{\mathbb{A}^2} \omega^8_\psi(\ell(y,z)i(h))\phi(\xi_1,\xi_2,0,0)\,dy\,dz.$$

To prove Theorem 2.1 it is enough to prove that

$$\int_{\mathbb{A}^2} \omega^8_\psi(\ell(y,z)i(h))\phi(r_1,r_2,0,0)\,dy\,dz = \omega^4_\psi(s^3(h))\phi'(r_1,r_2) \tag{2.8}$$

for all $h \in \mathrm{SL}_2(\mathbb{A})$ and $r_1, r_2 \in \mathbb{A}$. Since SL_2 is generated by $\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ it is enough to check the equality (2.8) for these matrices only.

We start with the unipotent element. From the embedding of $\mathrm{SL}_2 \times \mathrm{SO}_4$ in $\widetilde{\mathrm{Sp}}_8$ we compute that

$$i\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} = \begin{pmatrix} I_4 & xI_4 \\ & I_4 \end{pmatrix}\begin{pmatrix} n(x) & \\ & n(x)^* \end{pmatrix}$$

where I_4 denotes the 4×4 identity matrix, $n(x) = \begin{pmatrix} 1 & x & x & -x^2 \\ & 1 & 0 & -x \\ & & 1 & -x \\ & & & 1 \end{pmatrix}$ and $n(x)^*$ is defined so that the above matrix is in $\widetilde{\mathrm{Sp}}_8$. In the left-hand side of (2.8) we may move $i(h)$ to the left (it normalizes $\ell(y,z)$). Thus

$$\int_{\mathbb{A}^2} \omega^8_\psi\left(i\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\ell(y,z)\right)\phi(r_1,r_2,0,0)\,dy\,dz =$$
$$\int_{\mathbb{A}^2} \omega^8_\psi\left(\begin{pmatrix} I_4 & xI_4 \\ & I_4 \end{pmatrix}\begin{pmatrix} n(x) & \\ & n(x)^* \end{pmatrix}\ell(y,z)\right)\phi(r_1,r_2,0,0)\,dy\,dz.$$

Using the Weil representation action this equals

$$\int_{\mathbb{A}^2} \omega_\psi^8(\ell(y,z))\phi(r_1, r_2 + xr_1, xr_1, -x(xr_1 + r_2))\, dy\, dz.$$

Changing variables $y \mapsto y + \rho^2 xr_1$ and $z \mapsto z - x(xr_1 + r_2)$ and using the Weil representation action, we obtain

$$\begin{aligned}\int_{\mathbb{A}^2} \omega_\psi^8(\ell(y,z))\phi(r_1, r_2 + (1-\rho)xr_1, 0, 0)\psi(x^3r_1^2 + (1-\rho^2)x^2r_1r_2 - \rho^2xr_2^2)\, dy\, dz \\ =&\phi'(r_1, r_2 + (1-\rho)xr_1)\psi(x^3r_1^2 + (1-\rho^2)x^2r_1r_2 - \rho^2xr_2^2) \\ =&\omega_\psi^4\left(s^3\begin{pmatrix}1 & x\\ & 1\end{pmatrix}\right)\phi'(r_1, r_2),\end{aligned}$$

as claimed.

Next we check (2.8) for $w = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$. We have

$$\int_{\mathbb{A}^2} \omega_\psi^8(\ell(y,z)i(w))\phi(r_1, r_2, 0, 0)\, dy\, dz = \int_{\mathbb{A}^2} \omega_\psi^8(i(w)\ell(y,z))\phi(r_1, r_2, 0, 0)\, dy\, dz.$$

Notice that $i(w)$ is the long Weyl element in $\widetilde{\mathrm{Sp}}_8$. Hence ω_ψ^8 acts by the Fourier transform, and we obtain

$$\begin{aligned}&\int_{\mathbb{A}^2}\int_{\mathbb{A}^4} \omega_\psi^8(\ell(y,z))\phi(u_1, u_2, u_3, u_4)\psi(r_1u_1 + r_2u_2)\, du_i\, dy\, dz \\ &= \int_{\mathbb{A}^2}\int_{\mathbb{A}^4} \omega_\psi^8(\ell(0,z))\phi(u_1, u_2 - \rho^2y, u_3 - \rho y, u_4)\psi(r_1u_1 + r_2u_2 - yu_4)\, du_i\, dy\, dz.\end{aligned}$$

Changing variables this equals

$$\int_{\mathbb{A}^2}\int_{\mathbb{A}^4} \omega_\psi^8(\ell(0,z))\phi(u_1, u_2, u_3, u_4)\psi(r_1u_1 + r_2u_2 + y(\rho^2r_2 - u_4))\, du_i\, dy\, dz.$$

Given $f \in \mathcal{S}(\mathbb{A})$ we have

$$f(0) = \int_{\mathbb{A}}\int_{\mathbb{A}} f(u)\psi(yu)\, du\, dy.$$

Hence we obtain

$$\int_{\mathbb{A}}\int_{\mathbb{A}^3} \omega_\psi^8(\ell(0,z))\phi(u_1, u_2, u_3, \rho^2r_2)\psi(r_1u_1 + r_2u_2)\, du_1\, du_2\, du_3\, dz.$$

Using the action of the Weil representation and a change of variables $u_3 \mapsto -\rho u_3$ and $u_2 \mapsto u_2 - \rho^2u_3$ and $z \mapsto z + \rho^2r_2$ we then obtain

$$\begin{aligned}&\int_{\mathbb{A}}\int_{\mathbb{A}^3} \omega_\psi^8(\ell(u_3,z))\phi(u_1, u_2, 0, 0)\psi(r_1u_1 + (1+\rho)r_2u_2) du_1\, du_2\, du_3\, dz \\ &\qquad = \int_{\mathbb{A}^2} \phi'(u_1, u_2)\psi(r_1u_1 - \rho^2r_2u_2)\, du_1\, du_2 = \omega_\psi^4(s^3(w))\phi'(r_1, r_2).\end{aligned}$$

This completes the proof of Theorem 2.1. ∎

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